Two-Dimensional Motion Worksheet

Because perpendicular vectors are independent of each other we can use the kinematic equations to analyze the vertical ($y$) and horizontal ($x$) components of motion separately. This allows us to determine the position of the object as well as its vertical and horizontal velocity vectors at any time during the object’s flight.

Vertical Component

<table>
<thead>
<tr>
<th>Velocity</th>
<th>$v_y = a_y t + v_{0y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>$y - y_0 = \frac{1}{2}a_y t^2 + v_{0y} t$</td>
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Horizontal Component

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We can use the following vector equations to determine the components of acceleration and initial velocity:

$v_{0y} = v_0 \sin \theta$

$a_y = a \sin \theta$

$v_{0x} = v_0 \cos \theta$

$a_x = a \cos \theta$

And we can use the vector equations below to determine the object’s resultant velocity at any point during its flight:

$v = \sqrt{v_y^2 + v_x^2}$

$\theta = \tan^{-1} \frac{v_y}{v_x}$

We can use these equations to analyze both projectile motion and non-projectile motion. Here are the key characteristics of the two types of motions:

<table>
<thead>
<tr>
<th>Projectile Motion</th>
<th>Non-Projectile Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical Component</td>
<td>$a_y = -9.81 \text{ m/s}^2$</td>
</tr>
<tr>
<td>launched upward</td>
<td>$v_{0y} &gt; 0$</td>
</tr>
<tr>
<td>launched horizontally</td>
<td>$v_{0y} = 0$</td>
</tr>
<tr>
<td>launched downward</td>
<td>$v_{0y} &lt; 0$</td>
</tr>
</tbody>
</table>

| Horizontal Component | $a_x = 0$ |
|----------------------| $v_{0x} \neq 0$ |
| $a_x \neq 0$ | $v_{0x} \neq 0$ |

We’ll solve the following types of problems:

- General Trajectory – determine the position, component velocities and resultant velocity at any time during the flight.
- Flight Path – create an x-y graph showing the path of the object.
- Maximum Height – determine the distance between the initial launch point and the peak of the flight.
- Will it clear the fence? – determine if the projectile will clear an obstacle of known height located a known horizontal distance from the launch point.
- Where will it land? – determine the horizontal displacement (distance) the object travels from the launch point to the point it lands. (Projectile motion only.)
- Time of Flight – determine how long a projectile is in the air to reach a specified height. (Projectile motion only.)
**General Trajectory** – We can use the component displacement and velocity equations to describe the motion of an object at any point along its path of motion.

Example – An archer shoots an arrow at a trajectory (launch angle) of 25° with an initial velocity of 95 m/s. Let the origin be the point where the arrow left the bow. Find the position and velocity of the arrow at 0.5 second intervals over the first 3 seconds of its flight.

**Solution**

determine position using the component displacement equations, \(x_0 = 0\) meters and \(v_0 = 0\) meters.

at \(t = 0.5\) seconds,

\[
y - y_0 = \frac{1}{2}a_y t^2 + v_{0y} t = y = \frac{1}{2}a_y t^2 + v_{0y} t + y_0
\]

\[
y = \frac{1}{2}(-9.81 \text{ m/s}^2)(0.5\text{s})^2 + (95 \text{ m/s} \sin 25^\circ)0.5\text{s} + 0 = 19\text{ m}
\]

\[
x - x_0 = \frac{1}{2}a_x t^2 + v_{0x} t = x = \frac{1}{2}a_x t^2 + v_{0x} t + x_0
\]

\[
x = \frac{1}{2}(0\text{ m/s}^2)(0.5\text{s})^2 + (95 \text{ m/s} \cos 25^\circ)0.5\text{s} + 0 = 43\text{ m}
\]

the arrow’s position at \(t = 0.5\text{s}\) is (43m, 19m)

determine the velocity using the component velocity equations, the resultant vector magnitude and direction equations, and \(v_0 = 95\) m/s

\[
v_y = a_y t + v_{0y} = (-9.81 \text{ m/s}^2)0.5\text{s} + 95 \text{ m/s} \sin 25^\circ = 35 \text{ m/s}
\]

\[
v_x = a_x t + v_{0x} = (0 \text{ m/s}^2)0.5\text{s} + 95 \text{ m/s} \cos 25^\circ = 86 \text{ m/s}
\]

\[
v = \sqrt{v_y^2 + v_x^2} = \sqrt{(35.2 \text{ m/s})^2 + (86.1 \text{ m/s})^2} = 93 \text{ m/s} \quad \text{and} \quad \theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{35.2}{86.1} = 22^\circ
\]

Note: The arctangent function does not yield a unique solution. The two possible solutions are the angle your calculator gives you and the angle that is 180° opposite of your calculator answer. You need to determine which angle represents the physical situation.

Perform these calculations for \(t = 1.0\text{s}, 1.5\text{s}, 2.0\text{s}, 2.5\text{s}\) and 3.0s to complete the problem. Note, at negative \(y\)-value means the arrow has dropped below its initial launch point. Depending on the physical situation this may or may not be a valid answer.

**Flight Path** – To create an x-y graph of the object’s flight, rearrange the x-component displacement equation to solve for time. Then substitute this expression for time in the y-component displacement. This gives you an equation that shows the y-component displacement as a function of the x-component displacement instead of time. You can graph this equation using Excel or a graphing calculator.

Example – Create the flight path graph for the arrow in the previous example problem.

**Solution**

rearrange the x-component displacement equation for time

\[
x - x_0 = \frac{1}{2}a_x t^2 + v_{0x} t = v_{0x} t \quad \Rightarrow \quad t = \frac{x - x_0}{v_{0x}} = \frac{x - x_0}{v_0 \cos \theta}, \text{ substituting for } t \text{ in the } y\text{-component equation}
\]

\[
y - y_0 = \frac{1}{2}a_y t^2 + v_{0y} t = \frac{1}{2}a_y \left(\frac{x - x_0}{v_0 \cos \theta}\right)^2 + v_0 \sin \theta \left(\frac{x - x_0}{v_0 \cos \theta}\right) \quad \Rightarrow y = \frac{1}{2}a_y \left(\frac{x - x_0}{v_0 \cos \theta}\right)^2 + \frac{\sin \theta}{\cos \theta} \left(x - x_0\right) + y_0
\]
inserting the values for \( a_y \), \( v_0 \), \( x_0 \), \( y_0 \), and \( \theta \) from the problem

\[
y = \frac{1}{2}(-9.81 \text{ m/s}^2) \left( \frac{x-0 \text{ m}}{95 \text{ m/s} \cos 25^\circ} \right)^2 + \frac{\sin 25^\circ}{\cos 25^\circ} (x - 0 \text{ m}) + 0m = -0.000662 \frac{1}{m} x^2 + 0.466x
\]

Graph this equation with your calculator or use the equation to set up a table of \( x-y \) values in Excel and graph them. When using Excel, be sure to change \( x \) in increments small enough to give you an accurate graph.

The direction the object is moving (angle of trajectory) is always changing in projectile motion. The direction the object is moving is the same as the slope of the flight path at any given \( x \)-value.

\[
slope \text{ of flight path} = \frac{dy}{dx} = f'(x) = 2(-0.000662 \frac{1}{m})x + 0.466
\]

Therefore, to determine the direction as an angle we need to take the arctangent of the slope.

\[
\theta = \tan^{-1}(-0.00132 \frac{1}{m} x + 0.466)
\]

Note: The arctangent function does not yield a unique solution. The two possible solutions are the angle your calculator gives you and the angle that is 180° opposite of your calculator answer. You need to determine which angle represents the physical situation.

**Maximum Height** – To determine the maximum height of an object’s unimpeded flight path, we need to use the fact that the vertical component of velocity equals zero at this point (\( v_y = 0 \)). Rearrange the \( y \)-component velocity equation to solve for time. Then, substitute this expression for time in the \( y \)-component displacement equation and solve for the peak height (\( y_{\text{max}} \)).

Example – What is the maximum height of the arrow in the original example problem?

Solution

Rearrange the vertical component velocity equation to solve for time,

\[
v_y = a_y t + v_{oy} \Rightarrow t = \frac{v_y - v_{oy}}{a_y} = \frac{0-0}{a_y} = \frac{-v_{oy}}{a_y}
\]

Substituting into the \( y \)-component displacement equation

\[
y - y_0 = \frac{1}{2}a_y t^2 + v_{oy} t = \frac{1}{2}a_y \left( \frac{-v_{oy}}{a_y} \right)^2 + v_{oy} \left( \frac{-v_{oy}}{a_y} \right) = \left( \frac{1}{2} \right) \frac{v_{oy}^2}{a_y} + \frac{-v_{oy}^2}{-2a_y} = \frac{v_{oy}^2}{-2a_y} - \frac{v_{oy}^2 \sin^2 \theta}{-2a_y}
\]

Rearranging to solve for maximum height, 

\[
y_{\text{max}} = \frac{v_{oy}^2 \sin^2 \theta}{-2a_y} + y_0
\]
inserting the values for \(a\), \(v_o\), \(y_o\), and \(\theta\) from the problem, \(y_{max} = \frac{(95\ m)^2 \sin^2 25^\circ}{-2\left(-9.81\ m/s^2\right)} + 0m = 82m\)

**Will it clear an obstacle?** – To determine if the object will clear an obstacle located a known horizontal distance from the launch point, use the Flight Path equation, \(y = f(x)\). Set \(x\) equal to the horizontal distance between the launch point and the obstacle, and set \(y_o\) equal to the height of the launch point above a fixed reference point, i.e. the ground. If \(y > y_o\) the height of the obstacle above the fixed reference point, then the object will clear the obstacle. If \(y < y_o\) the height of the obstacle above the fixed reference point, then the object will not clear the obstacle.

Example – The attacking army’s archers want to launch flaming arrows into their enemy’s castle to burn the interior. The archers are 550 meters from the castle wall, which is 35 meters tall. Their arrows leave their bows 1.5 meters above the ground at an angle of 25\(^\circ\) and an initial velocity of 95 m/s. Assume this battle is taking place on perfectly level terrain. Will the arrows clear the castle walls?

**Solution**

inserting the values for \(a\), \(v_o\), \(x_o\), \(y_o\), and \(\theta\) from the problem into the Flight Path equation,

\[
y = \frac{1}{2}a_y \left( \frac{x-x_0}{v_0 \cos \theta} \right)^2 + \frac{\sin \theta}{\cos \theta} \left( x - x_0 \right) + y_0 = \frac{1}{2}(-9.81\ m/s^2) \left( \frac{550m-0m}{95\ m/s \cos 25^\circ} \right)^2 + \frac{\sin 25^\circ}{\cos 25^\circ} \left( 550m - 0m \right) + 1.5m
\]

\[
y = 57.8m \Rightarrow 58m \quad \therefore \text{the arrows will clear the castle wall}
\]

**Where will it land?** – To determine where the arrow will land, we need to rearrange the \(y\)-component displacement equation to solve for time. This is a quadratic equation that will find two times when the object reached the specified value for \(y - y_o\). In this scenario \(y_o\) is the launch point height above a fixed reference point and \(y\) is the height of the landing zone above the reference point. If \(y > y_o\), then the landing zone is above the launch point. If \(y < y_o\), then the landing zone is below than the launch point. If \(y = y_o\), then the landing zone is at the same height as the launch point. We will substitute the times into the \(x\)-component displacement equation and find the two values for the horizontal displacement, \(x - x_0\). Usually only one of these solutions will fit the physical scenario described in the problem.

Example – In the previous problem, the archers’ arrows clear the castle wall by 23 meters. Will they land inside the castle where they can ignite flammable objects or will the arrows hit or clear the stone wall on the opposite side of the castle? The wall on the opposite side of the castle is also 35 meters high and 250 meters from the front wall.

**Solution**

We need to rearrange the \(y\)-component displacement equation to solve for time. Since there are \(t^2\) and \(t\) terms, this is a quadratic equation and we need to use the quadratic formula to solve for \(t\).

\[
y - y_0 = \frac{1}{2}a_y t^2 + v_{0y}t \Rightarrow 0 = \left( \frac{a_y}{2} \right) t^2 + v_0 (\sin \theta) t + (y_0 - y)
\]

solving the quadratic equation,

\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow t = \frac{-v_0 \sin \theta \pm \sqrt{(v_0 \sin \theta)^2 - 4 \left( \frac{a_y}{2} \right) (y_0 - y)}}{2 \left( \frac{a_y}{2} \right)} = \frac{-v_0 \sin \theta}{a_y} \pm \sqrt{\left( \frac{v_0 \sin \theta}{a_y} \right)^2 - \frac{2(y_0 - y)}{a_y}}
\]
substituting values from the problem

\[
t = \frac{-95 \text{m} \sin 25^\circ}{-9.81 \text{m/s}^2} \pm \sqrt{\left(\frac{95 \text{m}}{x} \sin 25^\circ\right)^2 - \frac{2(1.5 \text{m} - 0 \text{m})}{-9.81 \text{m/s}^2}} = 4.09 \text{s} \pm \sqrt{16.7 \text{s}^2 - (-0.306 \text{s}^2)} = 8.21 \text{s and } -0.0338 \text{s}
\]

insert the valid time(s) into the x-component displacement equation,

\[
x = x_0 = \frac{v_0^2 \cos \theta}{2} + v_{0x}t = v_0\cos \theta \cdot t = (95 \text{m/s} \cos 25^\circ) \cdot 8.21 \text{s} = 706.8 \text{m} \Rightarrow 710 \text{m}
\]

The interior of the castle has a horizontal displacement from the archers of 550m \( \leq x - x_0 \leq (550\text{m} + 250\text{m}) \), therefore, the arrows will land inside the castle. (Note that the second solution from the quadratic equation was negative time, which isn’t a valid solution.)

Special Case – when the launch point and landing point are at the same height the \( y_0 - y \) term is zero and the quadratic equation for time simplifies to

\[
t = \frac{-v_0 \sin \theta}{a_y} \pm \frac{\sqrt{(v_0 \sin \theta)^2 - 4\left(\frac{a_y}{2}\right)(y_0 - y)}}{\frac{a_y}{2}} = \frac{-v_0 \sin \theta}{a_y} \pm \sqrt{\left(\frac{v_0 \sin \theta}{a_y}\right)^2 - \frac{2(y_0 - y)}{a_y}} = 0 \text{s and } \frac{-2v_0 \sin \theta}{a_y},
\]

\( t = 0 \) represents the horizontal displacement at the launch point and is not a valid solution for determining where the object lands. Substituting the valid expression for time into the horizontal component displacement equation yields,

\[
x - x_0 = \frac{v_0^2 \cos \theta}{2} \sin 2 \theta = 2 \cos \theta \sin \theta \ y \text{ yields, } x - x_0 = \frac{(-v_0^2 \sin 2 \theta)}{a_y}
\]

Time of Flight – To determine how long a projectile is in motion when it reaches a specified height, we need to rearrange the vertical component displacement equation to solve for time. This was done in the previous section and yielded a quadratic equation with the following solution:

\[
t = \frac{-v_0 \sin \theta}{a_y} \pm \frac{\sqrt{(v_0 \sin \theta)^2 - 4\left(\frac{a_y}{2}\right)(y_0 - y)}}{\frac{a_y}{2}} = \frac{-v_0 \sin \theta}{a_y} \pm \sqrt{\left(\frac{v_0 \sin \theta}{a_y}\right)^2 - \frac{2(y_0 - y)}{a_y}}
\]

Note that this equation will not yield a real solution for values of \( y \) that exceed the maximum height. Also, this equation will yield negative values of \( t \), which are not valid solutions.

Example – In a previous problem we determined the archers’ arrows cleared the front wall of the castle at a height of 58m. How long did it take the arrows to reach the front wall of the castle?

Solution

In this problem, \( y = 58 \text{m} \) and \( y_0 = 1.5 \text{m} \). Substituting these values as well as those for \( \theta \) and \( v_0 \) into the equation,

\[
t = \frac{-v_0 \sin \theta}{a_y} \pm \frac{\sqrt{(v_0 \sin \theta)^2 - 2(y_0 - y)}}{\frac{a_y}{2}} = \frac{-95 \text{m} \sin 25^\circ}{-9.81 \text{m/s}^2} \pm \sqrt{\left(\frac{95 \text{m}}{x} \sin 25^\circ\right)^2 - \frac{2(1.5 \text{m} - 58 \text{m})}{-9.81 \text{m/s}^2}} = 6.4 \text{s and } 1.8 \text{s}
\]

There are two times when a projectile reaches a specified height, on the way up and on the way down. The only exception is at its maximum height, and then there is only one time. You need to pick the correct value based on your understanding of the physical situation. In this problem the arrows could not have covered the 550m distance to the castle wall in only 1.8 seconds, therefore 6.4 seconds is the correct answer.
Conceptual Questions

1. Why does a bowling ball move without acceleration when it rolls along a bowling alley?

2. In the absence of air resistance, why does the horizontal component of velocity for a projectile such as a bullet remain constant while the vertical component changes?

3. How does the downward component of projectile motion compare with free fall motion?

4. Use terms we learned about one dimensional motion to describe projectile motion:
   a. vertical component –
   b. horizontal component –

Math Applications

5. A ball is thrown horizontally at a height of 2.2 meters at a velocity of 65 m/s. Assume no air resistance.
   a. How long until the ball reaches the ground?

   b. How far did the ball travel horizontally when it hit the ground?
6. A person runs off a 5.0-meter high balcony in a horizontal direction at 5.0 m/s. The swimming pool is at ground level 5.5 meters away from the edge of the balcony. Will the person reach the pool? Assume no air resistance. Show all your calculations to support your answer.

7. Suppose the person leapt off the same balcony at the same speed but with a trajectory of 25° above horizontal. Would he or she reach the pool? Assume no air resistance. Show all your calculations to support your answer.

8. An artillery shell is fired 1.75m above the ground at a 30.0° angle and an initial velocity of 625 m/s. Assume no air resistance.
   a. What is the vertical component of the shell’s muzzle velocity?
   b. What is the horizontal component of the shell’s muzzle velocity?
   c. What was the shell’s maximum height?
d. How long until the shell reaches its peak (maximum height)?

e. How far did the shell travel horizontally when it hit the ground? Assume there were no obstacles in the way and the landing point is at the same elevation as the launch point.

f. The artillery unit is trying to hit an enemy target that is beyond a mountain that has an elevation of 1250 m above the artillery’s launch point and is located 31500 km from the launch point. Will the shell clear the mountain top?

9. A batter hits a baseball 1.0 meter above the ground at a 25° angle and an initial velocity of 45 m/s. Assume no air resistance.
   a. What is the ball’s height above the field and distance from home plate 1.0 seconds and 2.0 seconds after it was hit?
b. What is the ball’s velocity at 1.0 seconds and 2.0 seconds? Velocity is a vector quantity and you must determine the magnitude and direction of the ball.

c. The center field fence is 125 meters away and 3.0 meters high. Did the batter hit a homerun? Show your calculations. Assume the center fielder cannot catch the ball.

10. A batter pops up the ball directly toward second base. The ball left his bat 1.25 meters above the ground at an angle of 75° and an initial velocity of 25 m/s. The fielder catches the ball 1.25 meters above the ground. Ignore air resistance.
   a. How far from home plate was the fielder when he caught the ball?

   b. How high above the field did the ball travel?

   c. How long did the ball stay in the air before it was caught?
11. A marksman is trying to hit a target from 1610 meters away. The rifle barrel and target are the same height above the ground. The muzzle velocity of the rifle is 1250 m/s.
   a. At what angle should the shooter aim her rifle in order to hit the target? Assume there is no wind or air resistance.
   b. What is the maximum height of the bullet above the rifle barrel?
   c. How long does it take the bullet to reach the target?

12. A hunter fires her rifle at a deer horizontally (θ = 0°) 1.10 meters above the ground. The initial muzzle velocity is 975 m/s. She misses the deer.
   a. Assume she is on perfectly flat ground, there are no obstacles in the way, there is no wind, and we can ignore air resistance. How far will the bullet travel before it strikes the ground?
   b. What potential safety issue does this cause?

13. A jet fighter is flying horizontally 1500 meters above the ground at 250 m/s when it suddenly accelerates at a rate of 15.75 m/s² and an angle of 75°. What is its altitude and horizontal velocity 35 seconds later? (Note this is not projectile motion).